The role of Korteweg stresses in Geodynamics

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The role of Korteweg stresses in Geodynamics

i.e. the surface tension of diffusive interfaces

Free Energy formulation

- Cahn and Hilliard, 1958 proposed an expression for free energy at the interface between two fluids:

$$F_V = N_V \int_V [f_0(c) + k(\nabla c)^2] dV$$

uniform non-uniform

Free Energy of a Nonuniform System. I. Interfacial Free Energy

John W. Cahn and John E. Hilliard

General Electric Research Laboratory, Schenectady, New York

(Received July 29, 1957)

It is shown that the free energy of a volume V of an isotropic system of nonuniform composition or density is given by: $N_V \int_V [f_0(c) + \kappa(\nabla c)^2] dV$, where N_V is the number of molecules per unit volume, ∇c the composition or density gradient, f_0 the free energy per molecule of a homogeneous system, and κ a parameter which, in general, may be dependent on c and temperature, but for a regular solution is a constant which can be evaluated. This expression is used to determine the properties of a flat interface between two coexisting phases. In particular, we find that the thickness of the interface increases with increasing temperature and becomes infinite at the critical temperature T_c, and that at a temperature T just below T_c the interfacial free energy σ is proportional to $(T_c - T)^{3/2}$.

The predicted interfacial free energy and its temperature dependence are found to be in agreement with existing experimental data. The possibility of using optical measurements of the interface thickness to provide an additional check of our treatment is briefly discussed.

Free Energy formulation

- Cahn and Hilliard, 1958 proposed an expression for free energy at the interface between two fluids:

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Homogeneous fluid

Interface between two fluids

Free Energy formulation

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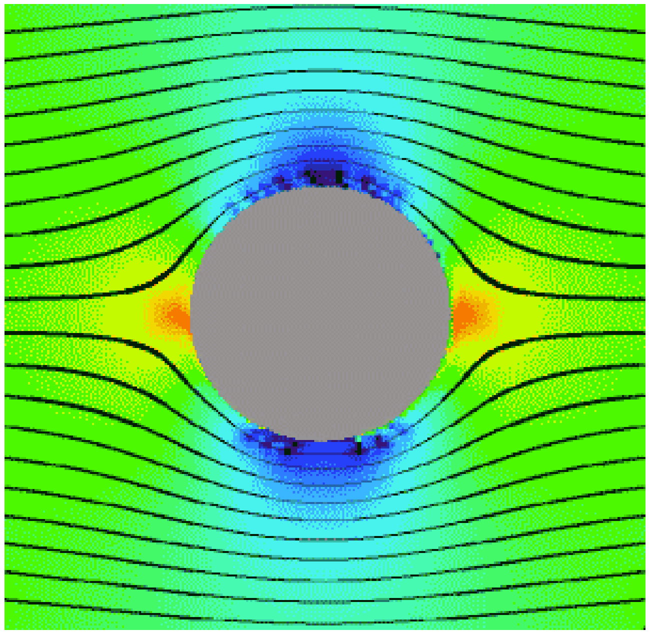
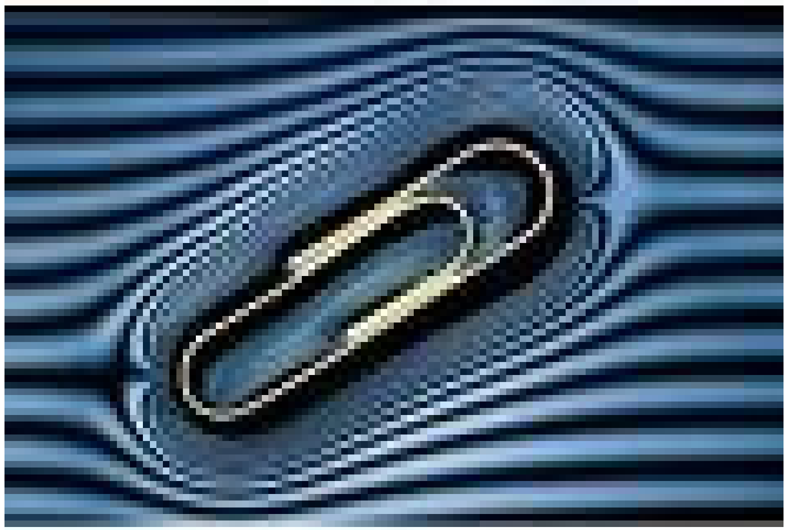
$$F_V = N_V \int_V [f_0(c) + k(\nabla c)^2] dV$$

Volume stresses



Surface tension



Free Energy formulation

- If we consider now the free energy for a system driven by gravity, the differential free energy between two fluids one over the other is the buoyancy plus a "surface" tension term:

$$F_V = N_V \int_V f_0(c)dV + F_S$$

$$f_0 = \Delta U + P\Delta V = k\Delta T + \alpha P\Delta T$$

$$F_{V0} = N_V \int_V [k\Delta T + \alpha P\Delta T]dV = N_V \int_V [\alpha P\Delta T]dV$$

This is the classical buoyancy term!

Free Energy formulation

- If we consider now the free energy for a system driven by gravity, the differential free energy between two fluids one over the other is the buoyancy plus a “surface” tension term:

$$F_V = N_V \int_V f_0(c)dV + F_S$$

$$F_S = N_V k \int_V (\nabla c)^2 dV$$

$$||\tau_{ij}|| \approx C \int_{-x_0}^{+x_0} (\partial_x \rho)^2 \, dx$$

This is the free surface energy, the surface tension!

Stokes flow

- Stokes flow represents the equilibrium between active (in general body) forces and viscous dissipation:

$$\partial_i \tau_{ij}^S + g_j = 0$$

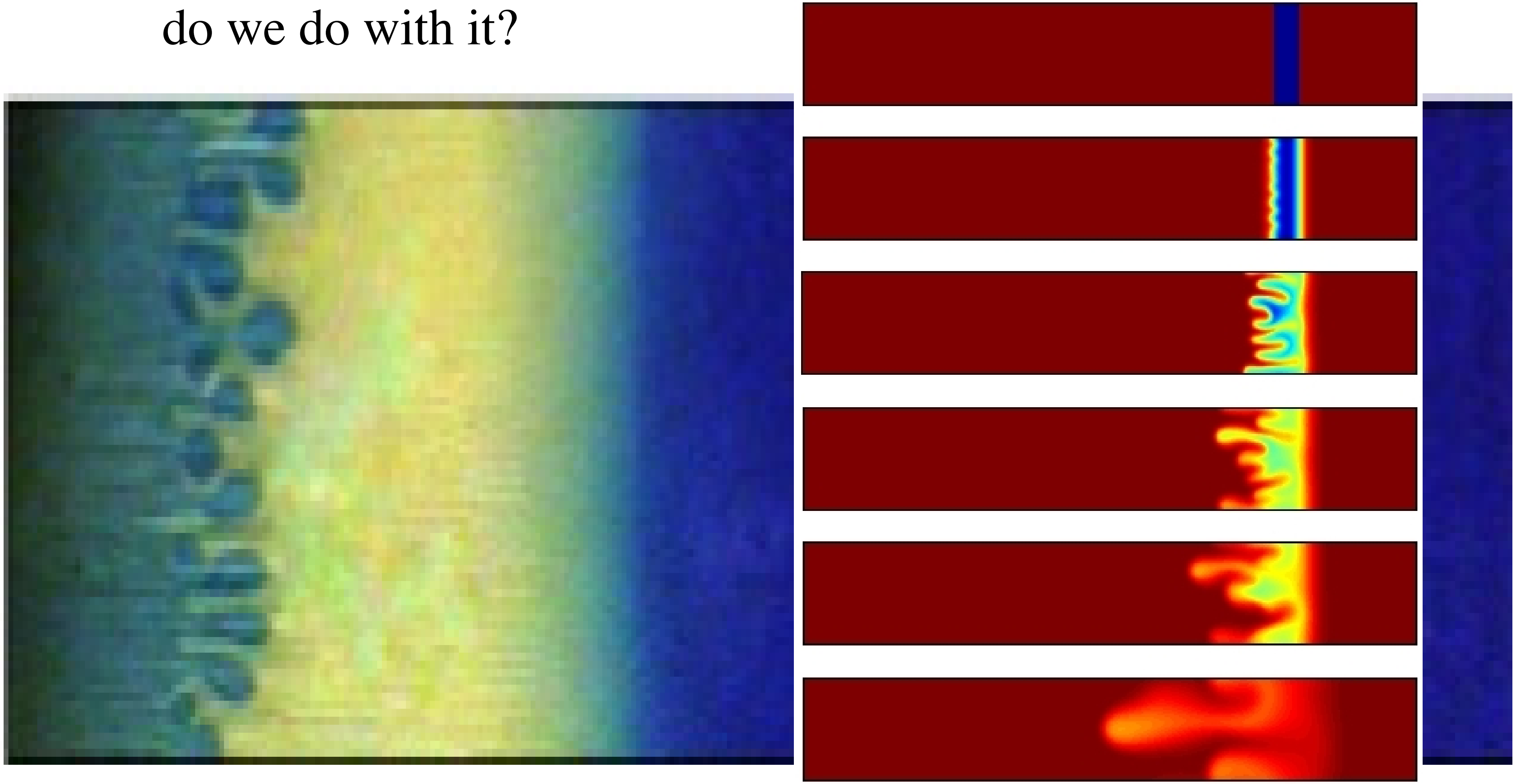
$$\tau_{ij}^S = \mu(\partial_i u_j + \partial_j u_i) - 2/3\delta_{ij}\partial_i u_i$$

- And surface tension? It is not in the stokes equation, but it is only applied to the boundary conditions and is rate-free:

$$||\tau_{ij}|| \approx C \int_{-x_0}^{+x_0} (\partial_x \rho)^2 \, dx$$

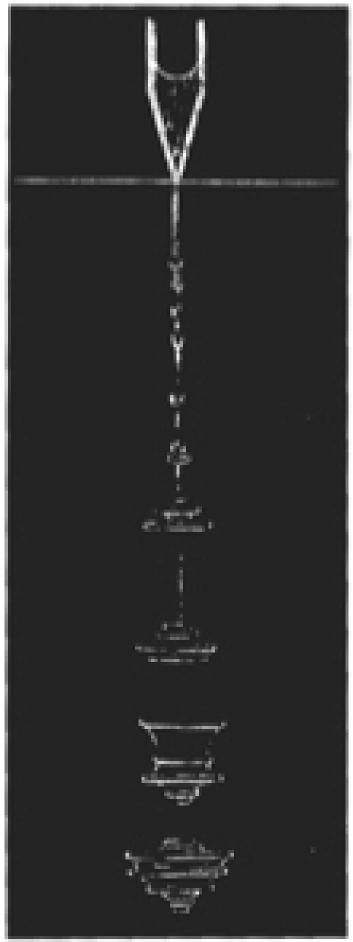
Miscible fluids

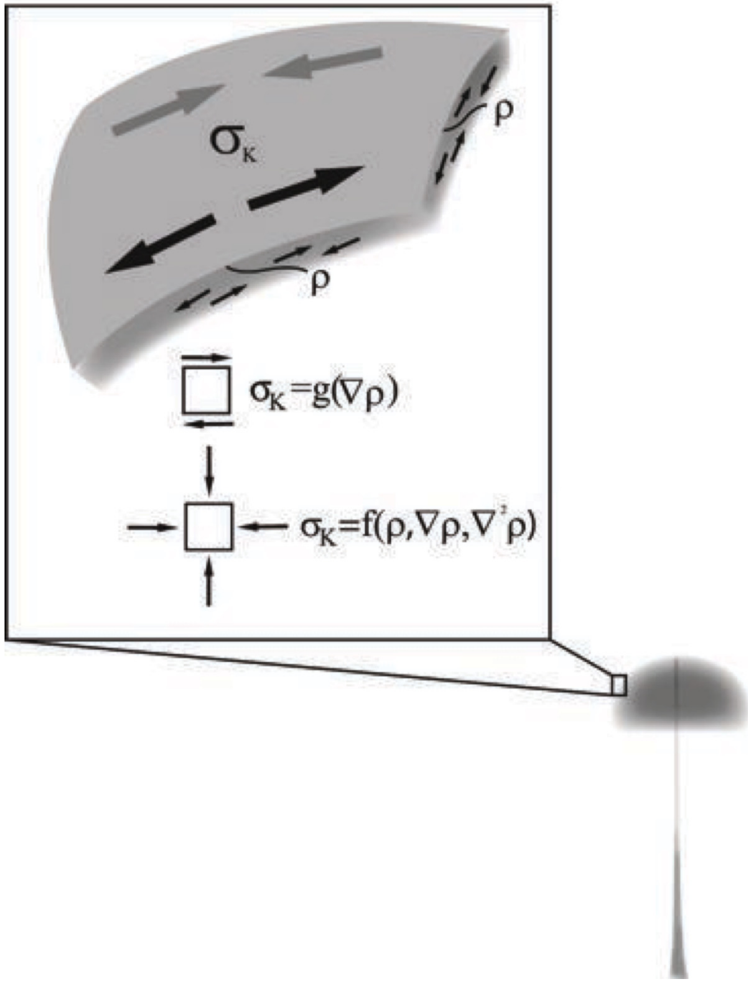
- But if the two fluids are miscible? Or if we have one fluid at two temperatures? Then we have a diffuse boundary! What do we do with it?



Korteweg stresses!

- At the miscible interface at the boundary of plume we will have “effective surface tensional stresses”:





Korteweg stresses

- Formally my Stokes equation changes and acquires an extra term that is proportional to the square of the density gradient

$$\partial_i \tau_{ij}^K + \partial_i \tau_{ij}^S + g_j = 0$$

$$\tau_{ij}^K = \delta_{ij}(\alpha \nabla^2 \rho + \beta \nabla \rho \nabla \rho) + \delta(\partial_i \rho)(\partial_j \rho) + \gamma \partial_i (\partial_j \rho)$$

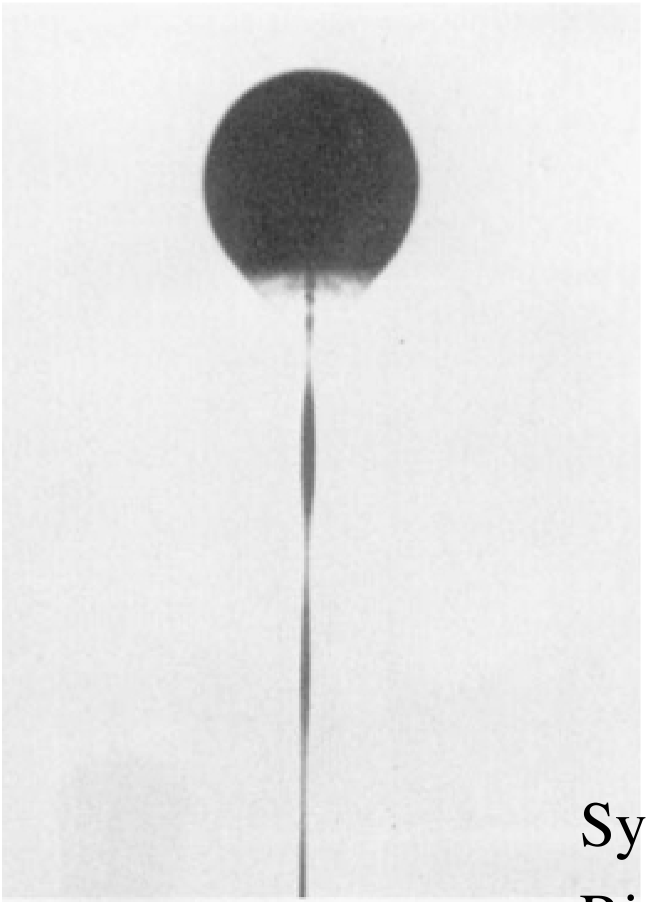
Only dependent on <u>**density**</u> gradients!

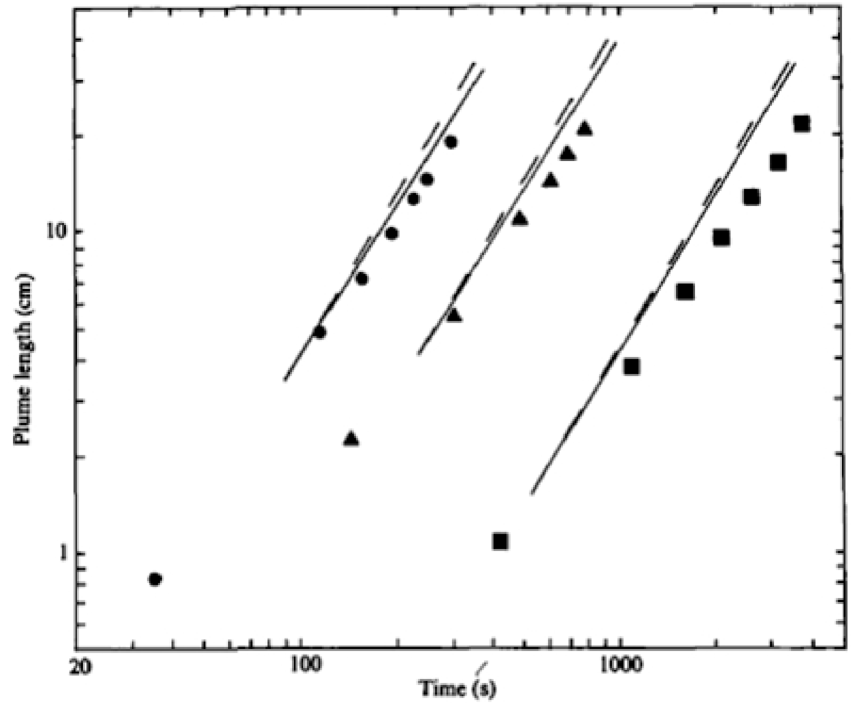
$$\tau_{ij}^S = \mu(\partial_i u_j + \partial_j u_i) - 2/3 \delta_{ij} \partial_i u_i$$

Only dependent on <u>**velocity**</u> gradients!

Korteweg stresses! Should we care?

- Experimental evidences: Olson and Singer, 1985



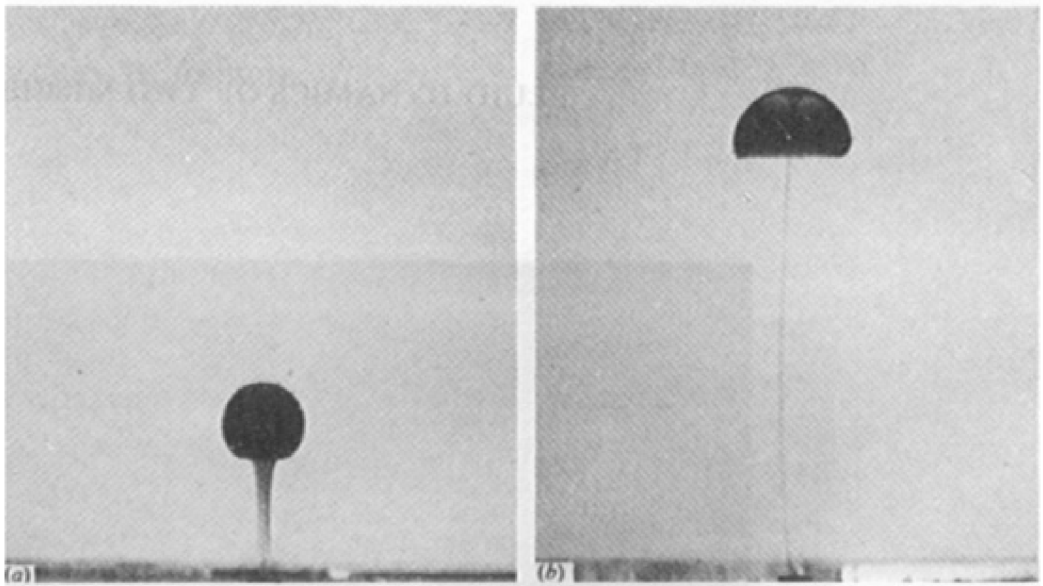


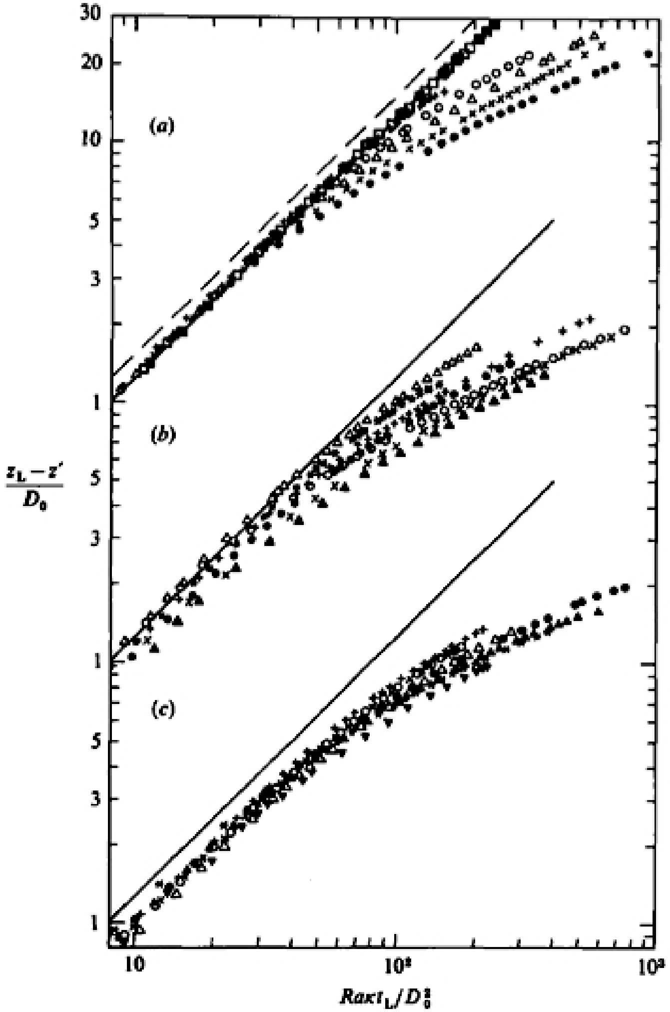
Systematic departure from Stokes law

Rising velocity as t^(2/5) instead of t^(2/3)

Korteweg stresses! Should we care?

- Experimental evidences:Griffiths, 1986





Rising velocity 22% lower then
expected Stokes velocity! Systematic.

Korteweg stresses! Should we care?

- Experimental evidences:

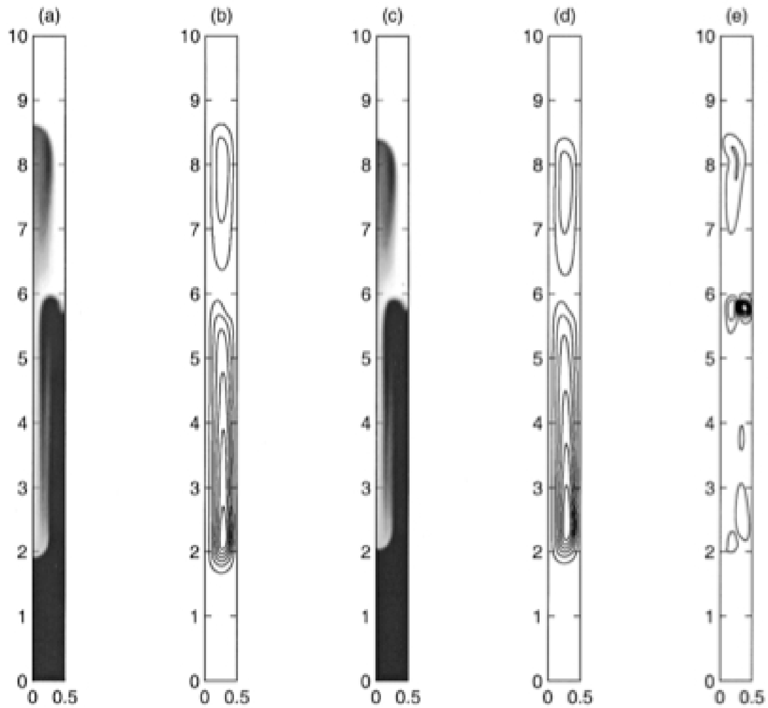
Laboratory: Petitjeanes and Maxworthy, 1996

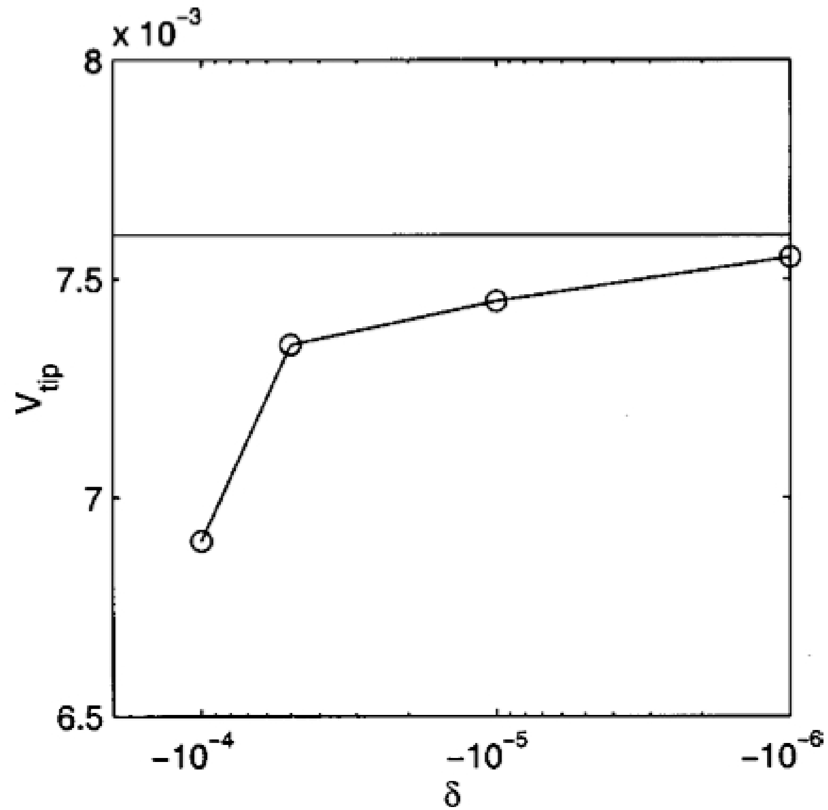
Numerical: Chen and Meiburg, 1996

Korteweg stresses! Should we care?

- Numerical evidences: Chen and Meiburg, 2001

$$\Delta P = \frac{1}{r}\sqrt{\frac{D}{t}}\left(\frac{k1\delta}{D} - k2\right)$$





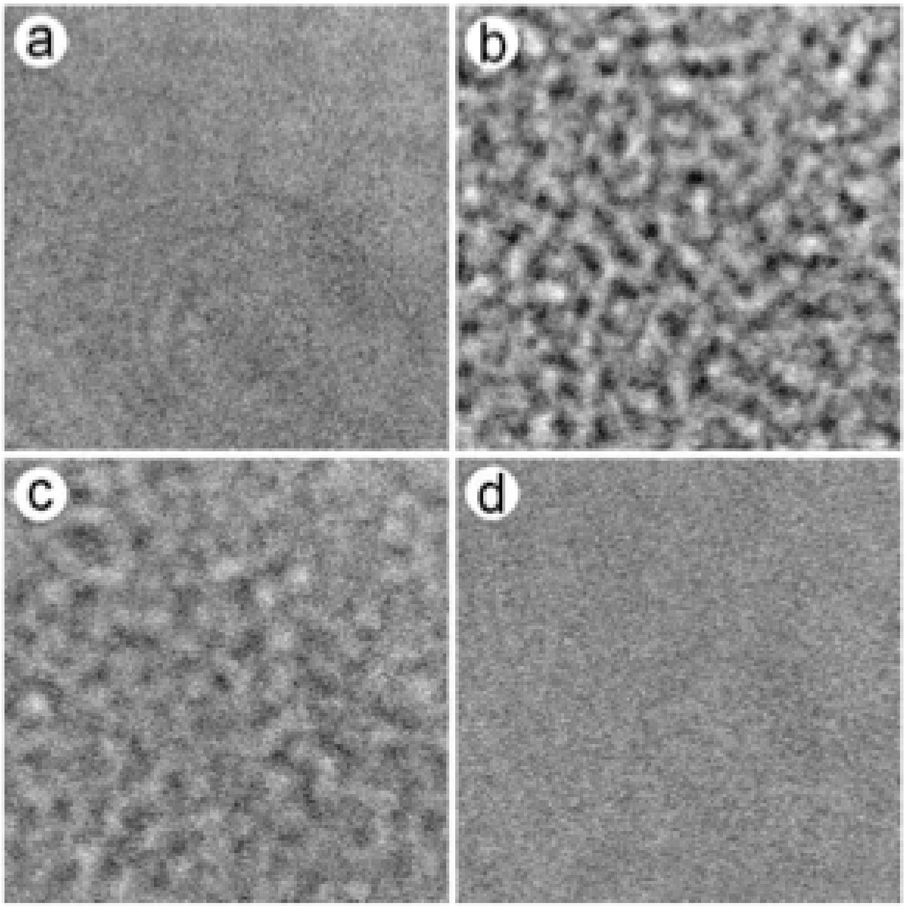
Korteweg stresses! Should we care?

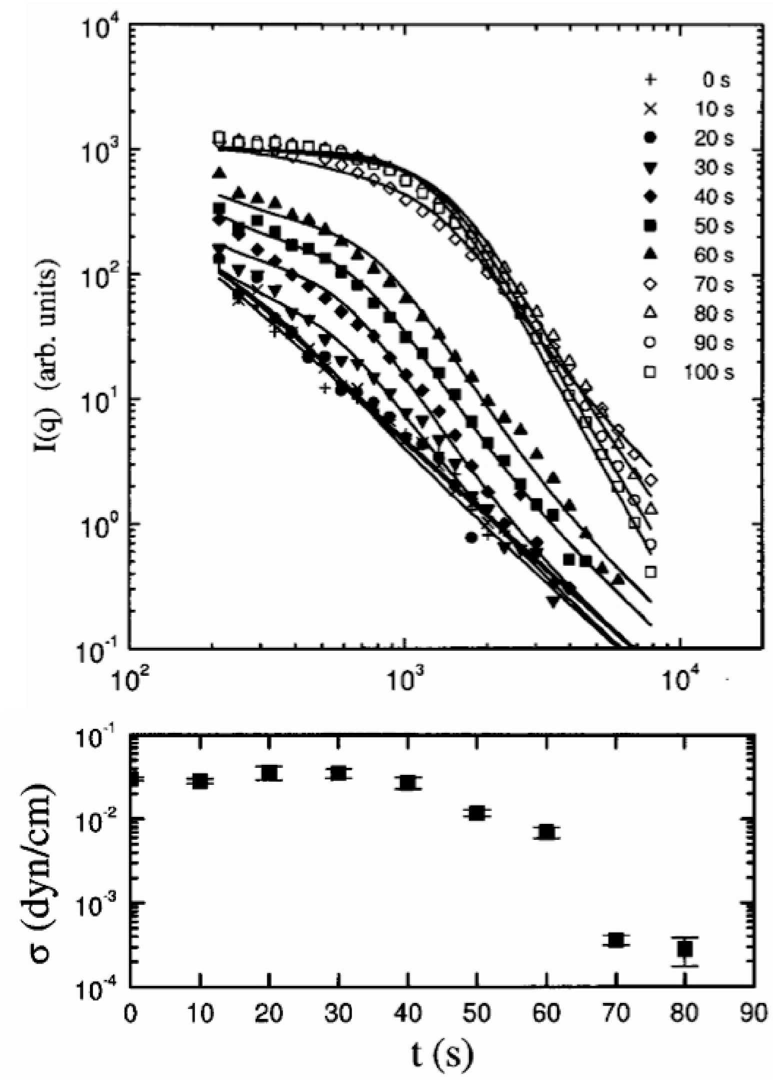
- Theoretical considerations:

- **Brenner (2005)** in his manuscript entitled "Navier Stokes Revisited" pointed out that the Navier Stokes equation requires to be modified in its application of the boundary conditions and also in the treatment of flows with density gradients

Korteweg stresses! Should we care?

- Indirect Evidence: Cicuta et al., 2000 & 2001)



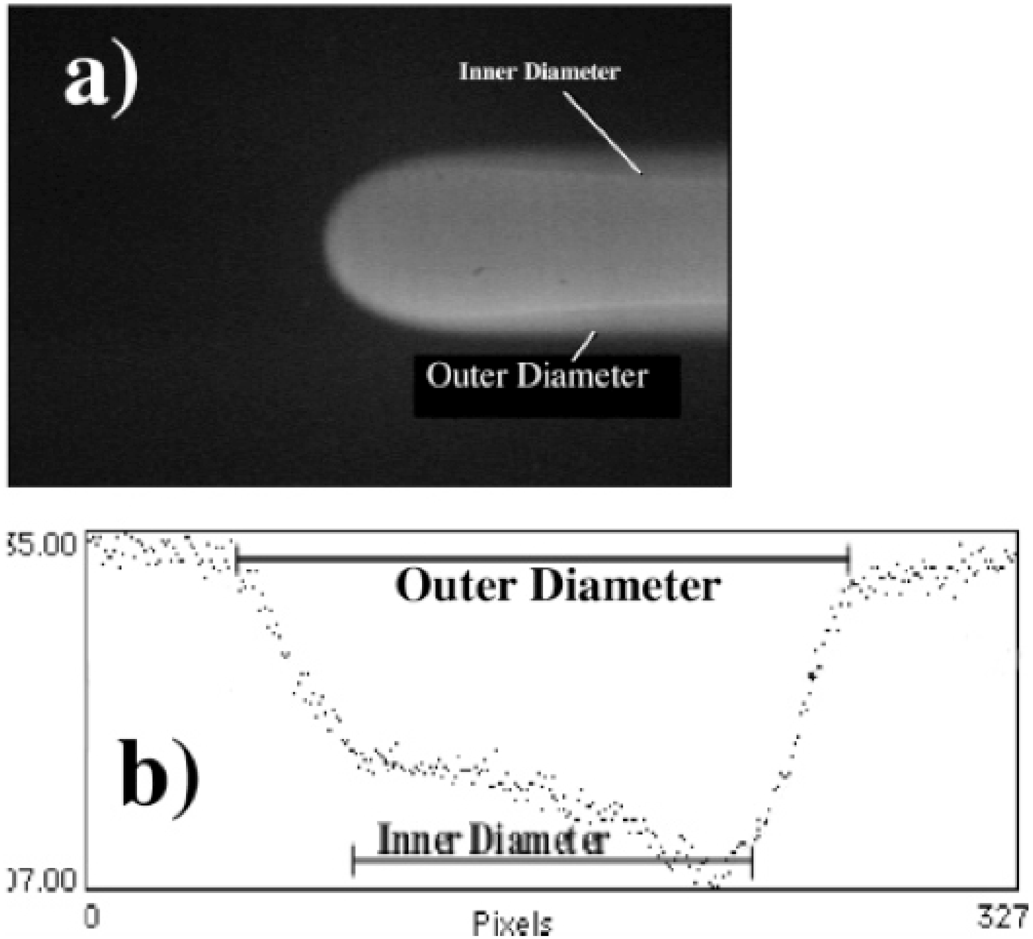


Korteweg stresses! Should we care?

- Direct Evidence: (Pojman, et al., 2006):

Interfacial tension = 0.6mN/m

$\delta = O(10^{-9}N)$



Consequences

- Feedback mechanisms can start at any scale. From the from the grain size (many of grain grow theories use the tension stresses of the grain boundaries) up to fluctuations at the meso scale (as in Cicuta et al.), until global geodynamics as plume velocity, phase transition (660km)

Consequences

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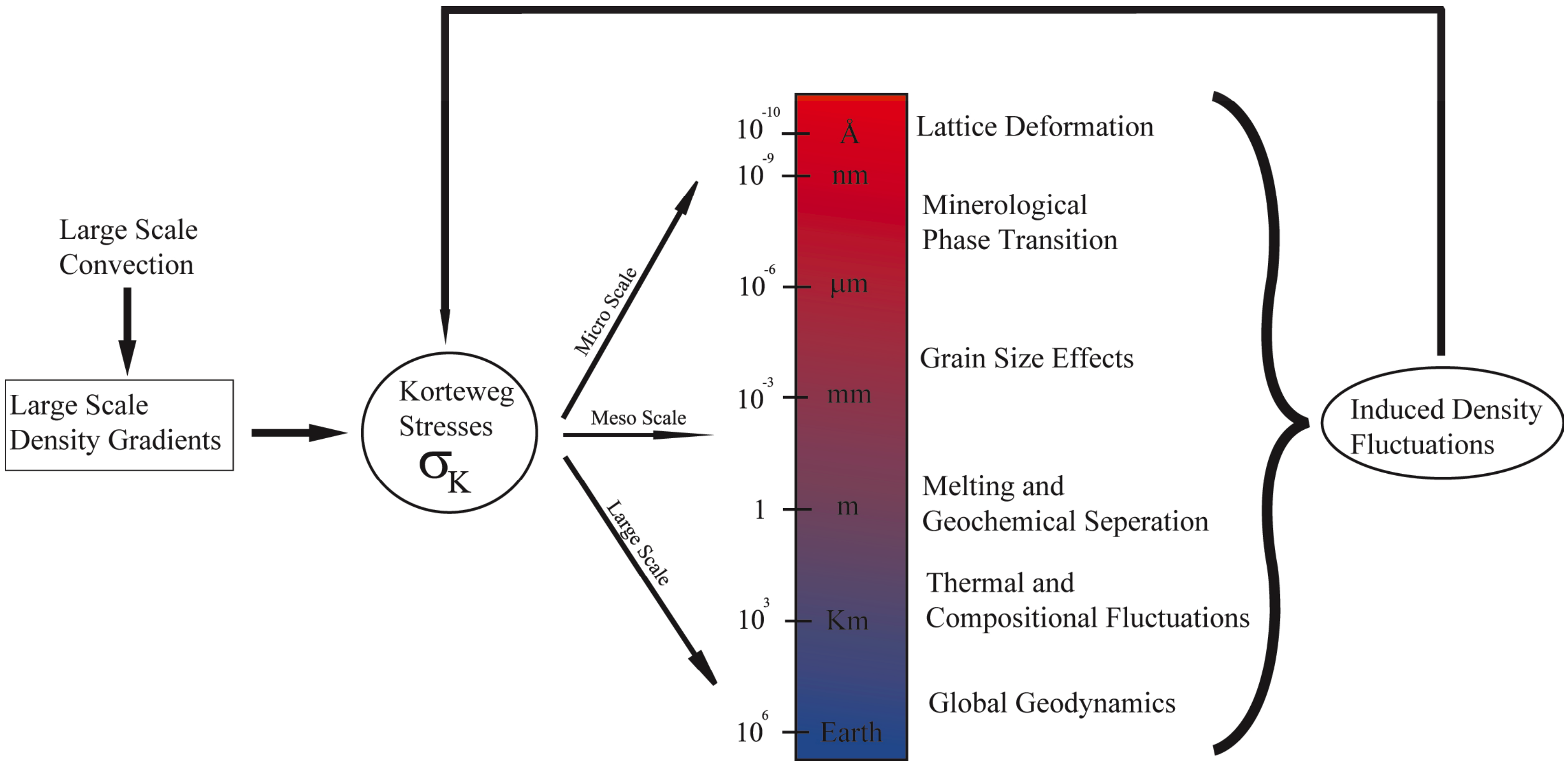
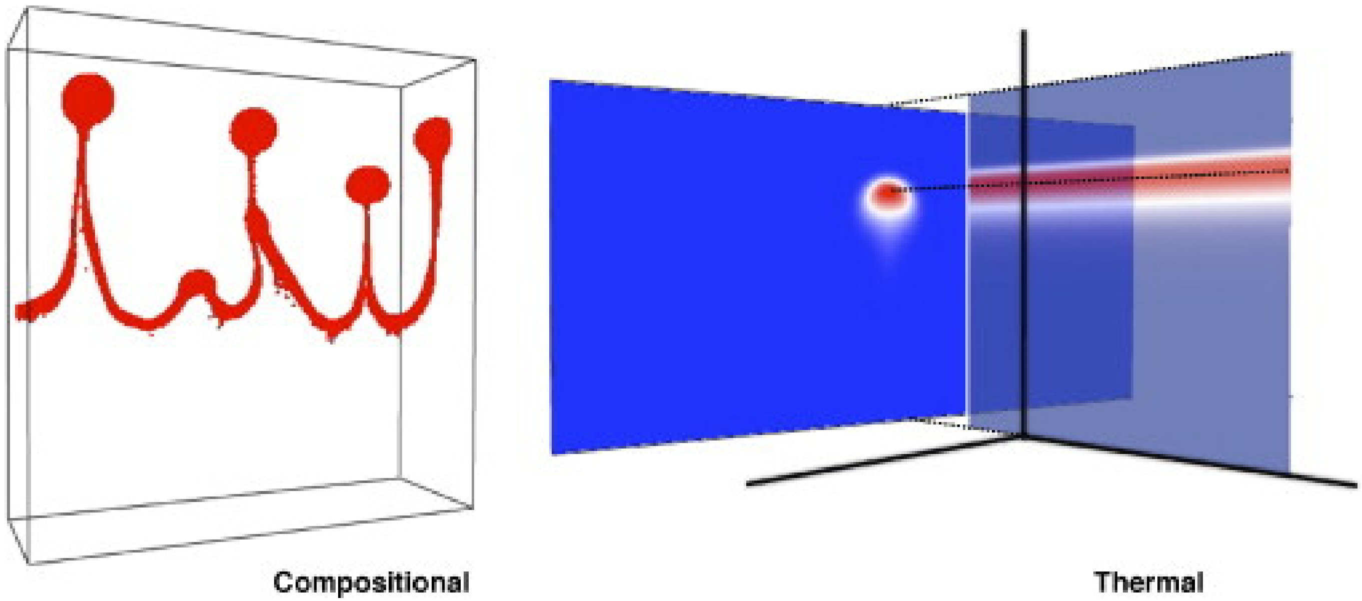


Consequences: e.g. thermal plume

- Simulation with Underworld. Thermal plume inhibits instabilities that are instead natural in compositional (non diffusive) plumes.



Quantifying Korteweg stresses

- How can we quantify Korteweg stresses for Earth?

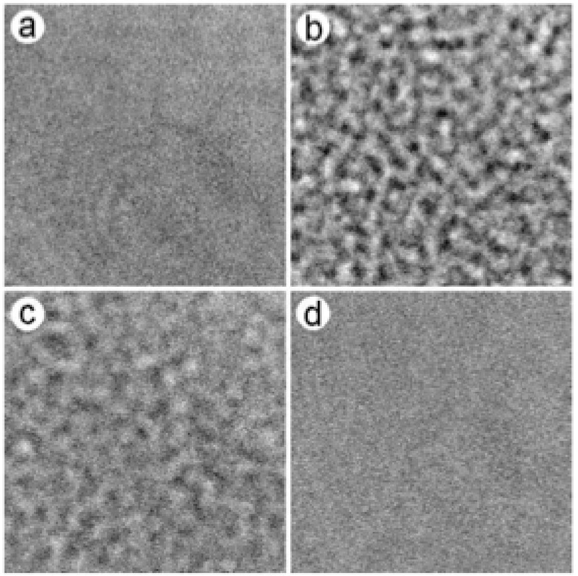
To be exact we need to cross the scales. Calculate stresses at the lattice scale and then having algorithms to upscale

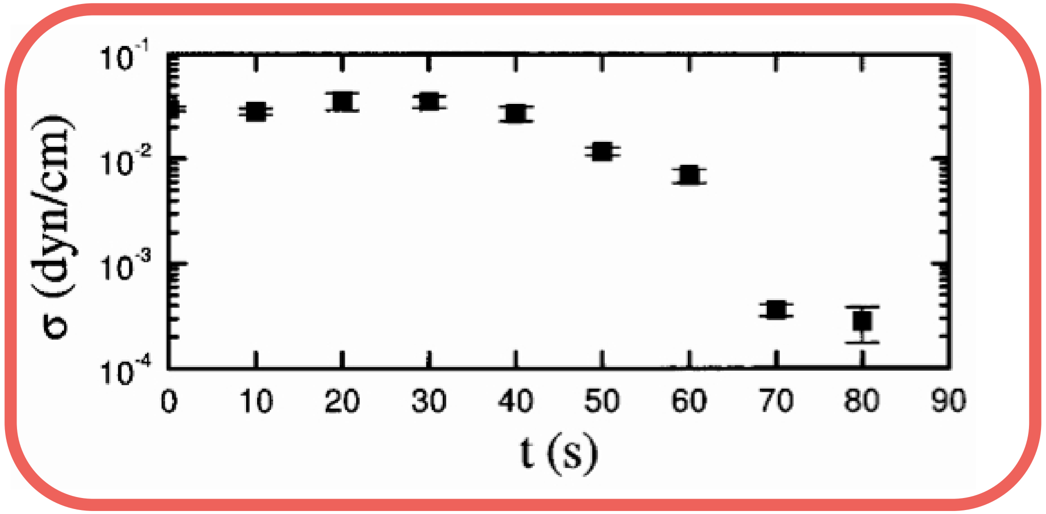
But before dong it we want to be sure that Korteweg stresses will be really important for geodynamics, so we can estimate them using some very general upper and lower bounds. Taking the work of (Cicuta et al, 2001), we get:

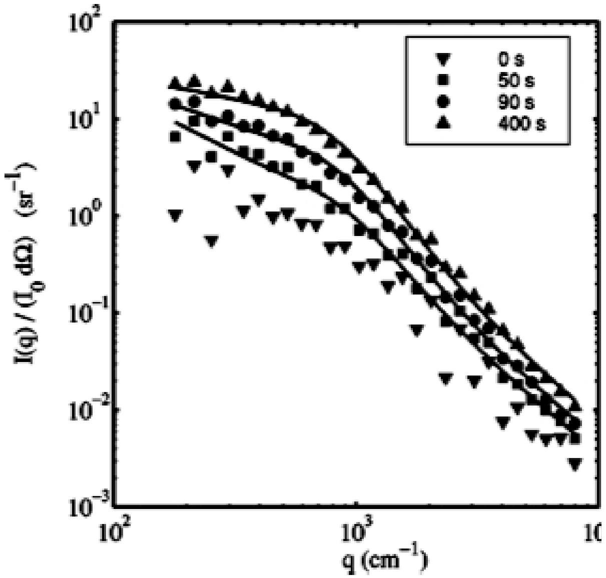
Roll-off wave vector (2pi/l): $$q_{RO} = \left(\frac{g \partial_z \rho}{\mu D} \right)^{1/4}$$

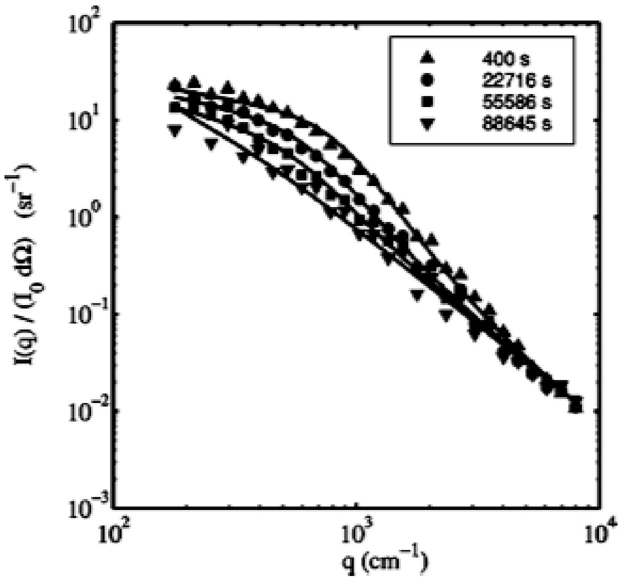
Roll-off capillary vector (2pi/l): $$q_{cap} = \left(\frac{g \Delta \rho}{\tau} \right)^{1/2}$$

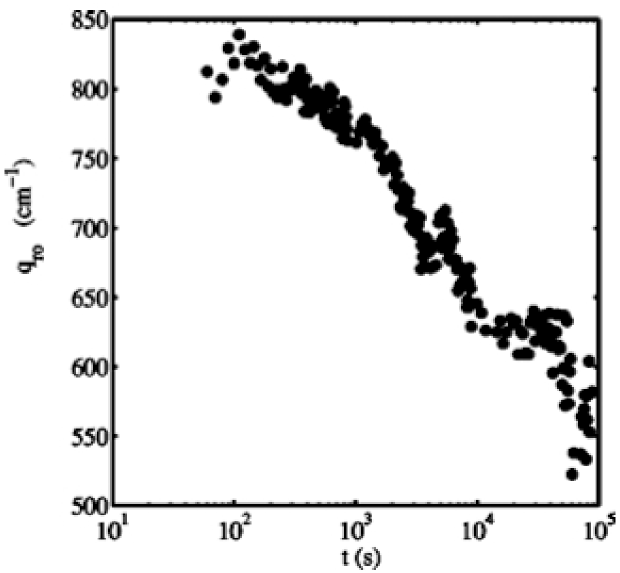
Quantifying Korteweg stresses









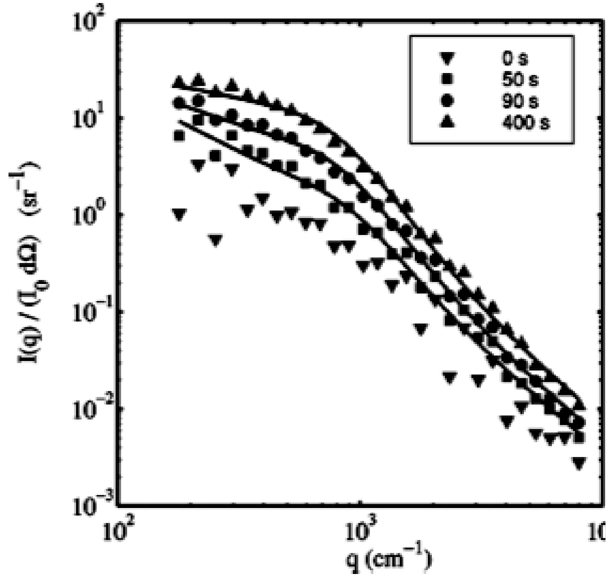


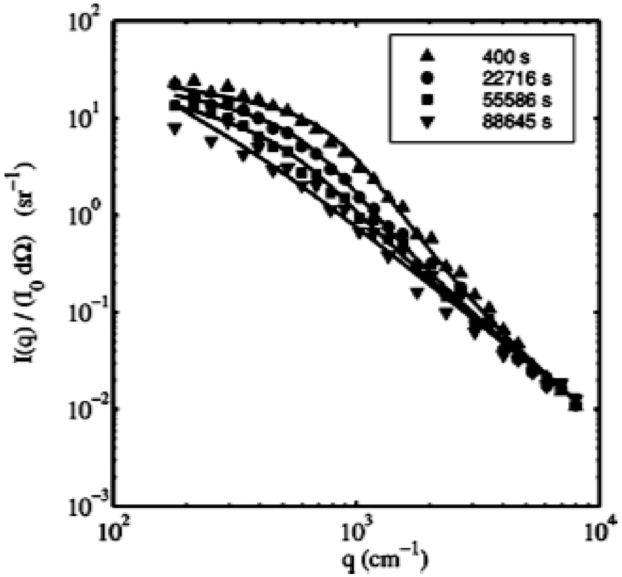
Quantifying Korteweg stresses

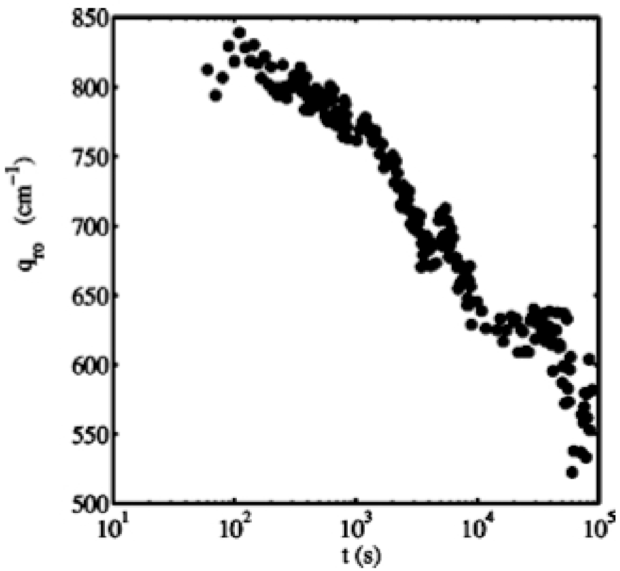
$$q_{cap} = \left(\frac{g\Delta\rho}{\tau}\right)^{1/2}$$

$$t_{co} = \frac{1}{Dq^2}\frac{1}{1+\left(\frac{q_{RO}}{q}\right)^4}$$

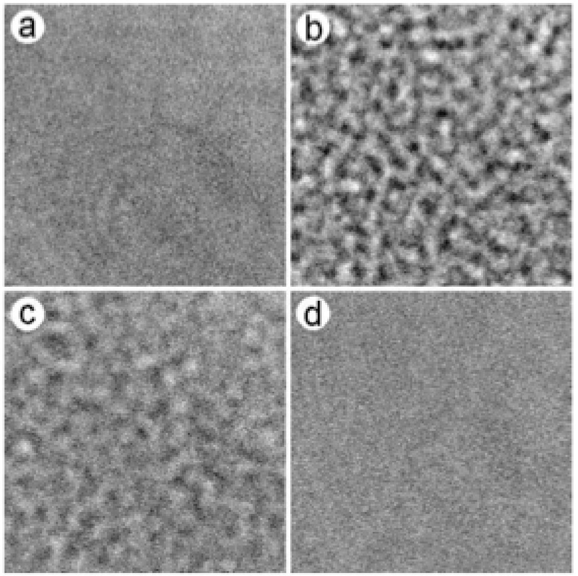
$$\tau = \frac{g\Delta\rho}{q_{cap}^2}$$







Quantifying Korteweg stresses

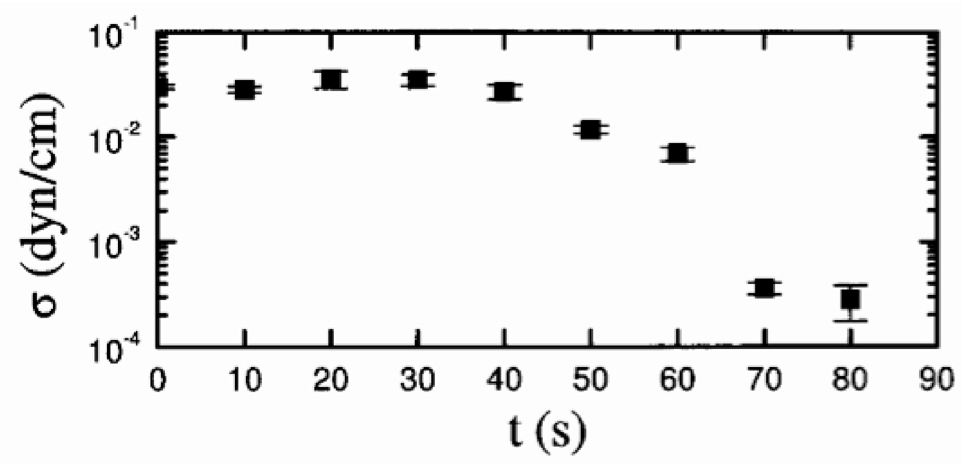


$$q_{RO} = \left(\frac{g \partial_z \rho}{\mu D} \right)^{1/4}$$

For lower mantle conditions for example, taking (g=10m/s^2, D=10^{-6}m^2/s, μ=10^{22}Pas), λ_{RO} oscillates between 200km for a sharp plume head boundary ($\Delta\rho$=100Kg/m^3 in Δz=10km) until 2000km for a very mild density fluctuation ($\Delta\rho$=1Kg/m^3 in Δx=1000 km).

Quantifying Korteweg stresses

$$t_{co} = \frac{1}{Dq^2}\frac{1}{1+\left(\frac{q_{RO}}{q}\right)^4}$$



t_{CO} value is extremely large under mantle conditions $O(10^{15}s)$ until $q>q_{RO}$, i.e. $\lambda<\lambda_{RO}$: beyond this threshold t_{co} decays extremely fast (forth power of q).
We can therefore estimate a lower bound for q_{cap}, being $q^{min}_{cap}=q_{RO}=2\pi/(200km)$.
An upper bound for q_{cap} can be instead set more simply considering the location of the steepest macroscopic gradient at the geodynamic scale, that we assume to be the boundary over the head of a plume $\Delta z=10km$, therefore $q^{max}_{cap}=2\pi/(10km)$.

Quantifying Korteweg stresses

$$q_{cap} = \left(\frac{g\Delta\rho}{\tau}\right)^{1/2}$$

$$\tau = \frac{g\Delta\rho}{q_{cap}^2}$$

Assuming $\Delta\rho$=100Kg/m^3, one finds $\tau_{min}\approx 2\cdot 10^9$ N/m and $\tau_{max}\approx 8\cdot 10^{11}$ N/m. For a the surface over the head of a plume with compositional thickness of 10km, and assuming a uniform shear stress through its all thickness, we get $\sigma_{min}\approx$200kpas and $\sigma_{max}\approx$80MPas, therefore an important contribution to the Korteweg stresses can be expected in the mantle

Conclusions

- Korteweg stresses are the equivalent of surface tension for diffusive fluids. There is a wealth of proofs that it exists.
- Analogue and numerical experiments put forward the idea that Korteweg stresses inhibit fingering and oppose low density dynamics
- Earth like plumes show a clear discrepancy from Stokes law that can be explained by Korteweg stresses
- Korteweg stresses have been directly observed at in mm scale experiments.
- Korteweg stresses require determining the konstants involved that depend on the multiscale merging of the free energies at the different scales.
- Momentum equations in geodynamics should be changed and the new term added.
- First estimations of Korteweg stresses for geodynamics brings to important values. They are expected to play an important role only close to the boundaries. But also large scale long wave density gradients might reveal important background effects.